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On Observable Subgroups of Complex Analytic Groups

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INTRODUCTION

Let L be a subgroup of a group G, and let ρ be a representation of L in a linear space V over a field F. A representation σ of G is said to be an *extension* of the representation ρ of L if the representation space W of σ contains V as an L-stable subspace and σ coincides with ρ in V. We say that a closed Lie subgroup L of a complex analytic group G is *observable* if every finite-dimensional analytic representation of L is extendable to a finite-dimensional analytic representation of G, or put another way, if every finite-dimensional analytic L-module is a sub L-module of a finite-dimensional analytic G-module. In this paper, we are interested in determining when a closed analytic subgroup of a faithfully representable complex analytic group is observable. For the similar question for (real) Lie groups, such a determination was made for normal subgroups [3, 6], and for the case of algebraic groups the observability question was satisfactorily solved in [1]. In Section 2, we extend some of the results in [1] to pro-affine algebraic groups, and in Section 3 we determine when the restriction map $R(G) \to R(L)$ induced by the inclusion $L \subset G$ is surjective. In order to describe our result in Section 4, we let L be a closed analytic subgroup of a faithfully representable complex analytic group and we express G as a semidirect product KH, where K is a closed, normal, simply connected solvable subgroup and H is a maximal reductive analytic subgroup of G. If N denotes the representation radical of G, then the closed normal subgroup HN of G contains all maximal reductive subgroups of G and has the property that its image under any finite-dimensional analytic linear representation of G is algebraic, and it is characterized as the largest normal subgroup of G with this property. The subgroup HN has a unique irreducible algebraic group structure which is compatible with its analytic group structure. We will call the subgroup NH the *algebraic kernel* of G. Our main result (Theorem 4.5) states that L is observable in G if and only if (i) the algebraic kernel of L is the intersection of L and the algebraic kernel of G, and (ii) the algebraic

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kernel of L is observable in the algebraic kernel of G. Our study relies heavily on the work of Bialyniccki–Birula, Hochschild, and Mostow on the subject, as well as on the general theory of representations and representative functions of Lie groups as developed in [2, 4].

Throughout this paper all subgroups of complex analytic groups are assumed to be complex Lie subgroups, and their representations are assumed to be complex analytic and finite-dimensional, unless stated otherwise.

1. Preliminaries

Let G be a complex Lie group. For an analytic finite-dimensional representation ρ of G and for any linear functional on the algebra of all C-linear endomorphisms of the representation space of ρ, the composite map $t \circ \rho$ is called a representative function on G associated with ρ. We denote by $[\rho]$ the C-linear space consisting of all representative functions on G that are associated with ρ. Let $R(G) = \cup\rho[\rho]$, where ρ runs over all finite-dimensional analytic representations of G. More generally, for a closed normal subgroup G_0 of G, let $R(G, G_0)$ denote the subalgebra of $R(G)$ consisting of all representative functions f on which the subgroup G_0 operates unipotently (that is, the representation of G by left translations on the space spanned by the left translates of f is unipotent on G_0). It is easy to see that the algebra $R(G, G_0) = \cup\rho[\rho]$, where ρ varies over the finite-dimensional analytic representation which is unipotent on G_0. A closed Lie subgroup L of G that contains G_0 induces the restriction morphism $R(G, G_0) \to R(L, G_0)$. We have:

LEMMA 1.1. *If an analytic representation ρ of L that is unipotent on G_0 extends to an analytic representation σ of G that is unipotent on G_0, then $[\rho]$ is contained in the image of the restriction map $R(G, G_0) \to R(L, G_0)$. In particular, if every analytic representation of L that is unipotent on G_0 extends to an analytic representation that is unipotent on G_0, then $R(G, G_0) \to R(L, G_0)$ is surjective.*

Proof. Let V be the representation space for ρ and let W be the representation space for σ. We may assume that W contains V as an L-invariant subspace. Choose a basis $w_1, w_2, \ldots, w_n$ of W such that $w_1, w_2, \ldots, w_m$ $(m \leq n)$ is a basis of V, and let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the dual basis of W°, the dual space of W. Then the restrictions $\gamma_i|_V$ $(1 \leq i \leq m)$ form the dual basis of V°, and the functions g_{ij}: $L \to C$, defined by $g_{ij}(y) = \gamma_i(\rho(y)(w_j))$, $1 \leq i \leq m$, span the space $[\rho]$. Since each g_{ij} is the restriction of the $f_{ij} \in [\sigma] \subset R(G, G_0)$, where f_{ij}: $G \to C$ is given by $f_{ij}(x) = \gamma_i(\sigma(x)(w_j))$, we see that $[\rho]$ is contained in the image of $R(G, G_0) \to R(L, G_0)$, proving the lemma.

Suppose now that G is a faithfully representable complex analytic group (i.e., G has a faithful finite-dimensional complex analytic representation or, equivalently, that the algebra $R(G)$ of all complex analytic representative functions on G separates the points of G). G has a semidirect product decomposition $G = H \cdot K$, where K is a simply connected solvable normal complex analytic subgroup of G and H is a maximal reductive subgroup of G [2, Theorem 9.1]. The group of all proper automorphisms of $R(G)$ (i.e., of algebra automorphisms of $R(G)$ that commute with every right translation $f \to f \cdot x$ effected by an element x of G on $R(G)$) is denoted by $A = A(G)$. There is a canonical bijection between A and $\mathrm{Spec}(R(G), C)$, and hence the group A has the structure of a pro-affine algebraic group. The canonical map $G \to A$ that associates with every element x of G the left translation $f \to x \cdot f$ effected by x on $R(G)$ is a monomorphism, and we identify G with its canonical image whenever convenient. The canonical map $G \to A$ satisfies the following "universal" property: For any finite-dimensional analytic representation ρ: $G \to GL(V)$, there exists a unique rational representation ρ^*: $A \to GL(V)$, which extends ρ. Every left G-stable subspace S of $R(G)$ is also A-stable, and we will denote by A_S or G_S the restriction image of A or G in the group of all C-linear automorphisms of S.

A subset of $R(G)$ is said to be *fully stable* if it is stable under the right and left G-translations as well as under the involution $f \to f'$, where $f(x) = f(x^{-1})$. If S is a finitely generated fully stable subalgebra of $R(G)$, then A_S is the group of all proper automorphisms of S and has a natural structure of an irreducible affine algebraic group with S as the algebra of all polynomial functions, where an element f of S is regarded as a C-valued function on A_S by $f(\alpha) = \alpha(f)(1)$. Thus structure of A_S defines also the structure of a complex analytic group on A_S. The subgroup G_S of A_S is a complex analytic subgroup of A_S, and the canonical map $G \to G_S$ is an epimorphism of complex analytic groups. Moreover, G_S is algebraically dense in A_S and contains the commutator subgroup of A_S. If T is a finitely generated fully stable subalgebra of $R(G)$ such that $T \subset S$ then the restriction map $A_S \to A_T$ is an epimorphism of algebraic groups, and A is the projective limit of this system of algebraic group epimorphisms.

If Y is a left G-submodule of S such that S is the smallest fully stable subalgebra of $R(G)$ containing Y then the restriction map $A_S \to A_Y$ is an isomorphism irreducible affine algebraic groups.

If X is a subset of A, we will denote by $R(G)^X$ the algebra of all X-fixed elements of the left A-module $R(G)$. If Y is a subset of $R(G)$ then A^Y denotes the subgroup consisting of all elements of A that leave the elements of Y fixed. We will use the same notation in connection with

A_S. Finally, we denote by Y' the image Y under the involution $f \to f'$ of $R(G)$.

Let $G = A \cdot B$ (semidirect product) and, for $f\colon B \to C$ (resp., $f\colon A \to C$), define $f\colon G \to C$ by $f^+(ab) = f(b)$ (resp., $f^+(ab) = f(a)$) for $a \in A$ and $b \in B$. For $a \in A$, $\kappa(a)$ denotes the automorphism of B given by $b \to aba^{-1}$. For $g \in R(B)$, $a \in A$, and $b \in B$, we have

$$(b \cdot g)\kappa(a) = (aba^{-1}) \cdot (g\kappa(a)).$$

For a fixed a, and with b ranges over B, this shows that $g\kappa(a) \in R(B)$.

The following lemma is a reformulation of [2, Proposition 2.4] and will be used later in Section 3.

LEMMA 1.2. *Let $G = A \cdot B$ be a semidirect product of complex Lie groups with B normal in G, and let B_0 be a closed normal subgroup of G with $B_0 < B$. Let $R(G, B_0)_B$ denote the image of the restriction map $R(G, B_0) \to R(B, B_0)$. Then for $g \in R(B, B_0)$, the following are equivalent:*

(i) $g \in R(G, B_0)_B$,

(ii) $g\kappa(A) \subset R(B, B_0)$ *and the subspace spanned by $g\kappa(A)$ is finite-dimensional.*

Moreover, $(f, g) \to f^+ \cdot g^+$ induces an isomorphism $R(A) \otimes R(G, B_0)_B \cong R(G, B_0)$. In particular, $R(G, B_0) \cong R(A) \otimes R(B, B_0)$ canonically if and only if the restriction morphism $R(G, B_0) \to R(B, B_0)$ is surjective.

2.

The main goal of this section is to extend some of the results of [1] to pro-affine algebraic groups.

The proof of the following lemma is stated for algebraic linear groups G in [1, Theorem 1], but the proof carries over easily to pro-affine algebraic groups, or, more generally, for any category of groups, provided that the restriction morphism $R(G) \to R(L)$ is surjective.

LEMMA 2.1. *Let G be a pro-affine algebraic group, and let L be an algebraic subgroup of G. Suppose that, for every one-dimensional rational L-module that is contained as an L-submodule in a rational G-module, the dual L-module is also an L-submodule of rational G-module. Then L is observable in G.*

For any affine or pro-affine algebraic group G, let $P(G)$ denote the algebra of polynomial functions on G, and let $[D]$, for any integral domain

D, denote the field of fractions of D. The following is the generalization of [1, Theorem 3] to pro-affine algebraic groups. The proof given here is essentially the same as that of [1, Theorem 3], except for overcoming the difficulty in the absence of finite-dimensionality of the group G.

THEOREM 2.2. *Let G be an irreducible pro-affine algebraic group over a field F, and let L be an algebraic subgroup of G. Then L is observable in G if and only if* $[P(G)]^L = [P(G)^L]$.

Proof. Assume that L is observable. Clearly $[P(G)]^L \supseteq [P(G)^L]$. To show $[P(G)]^L \subseteq [P(G)^L]$, let $q \in [P(G)]^L$. To show $q \in [P(G)^L]$, it is enough to show that $(P(G) \cdot q \cap P(G))^L \neq (0)$. Let V be any nonzero simple rational L-submodule $P(G) \cdot q \cap P(G)$, and let V° denote the dual L-module. Since L is observable, V° is a sub L-module of some finite-dimensional rational G-module W. Since W is isomorphic with a sub G-module of a direct sum of finitely many copies of $P(G)$ [2, Proposition 2.3], and since V° is simple, there is a monomorphism $\phi\colon V^\circ \to P(G)$. We choose a basis $v_1, \ldots, v_n$ of V such that $v_1(1) \neq 0$ and $v_i(1) = 0$ for $i = 2, \ldots, n$. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the dual basis of V°, and put $g_i = \phi(\gamma_i)$, for $1 \leq i \leq n$. Then, for every $x \in G$, the element $h = \sum_{i=1}^n (g_i x) v_i$ belongs to $(P(G) \cdot q \cap P(G))^L$. Now take x such that $g_1(x) \neq 0$. Then $h(1) = \sum_{i=1}^n (g_i x)(1) v_i(1) = (g_1 \cdot x)(1) = g_1(x) \neq 0$, and hence $h \neq 0$.

To prove the sufficiency of the condition, we note that, using a standard argument as used in the proof of [1, Theorem 3], we may reduce the assertion to the case in which the ground field F is algebraically closed. Thus we assume that F is algebraically closed and that $[P(G)]^L = [P(G)^L]$ to show the condition of Lemma 2.1 above. This condition is equivalent to showing the following: For any non-zero $f \in P(G)$ such that, for every $y \in L$, $y \cdot f = \gamma(y) f$ with $\gamma(y) \in F$, there exists a non-zero $g \in P(G)$ such that $y \cdot g = \gamma(y)^{-1} g$, for every $y \in L$.

For every $x \in G$, the fraction $(f \cdot x)/f \in [P(G)]^L$. The set $\{f \cdot x \mid x \in G\}$ spans a finite-dimensional space, and hence we can find elements $x_1, x_2, \ldots, x_n \in G$ so that $f \cdot x_1, f \cdot x_2, \ldots, f \cdot x_n$ is a basis for the C-linear space spanned by the $f \cdot x$, $x \in G$. For each $x \in G$, we have $f \cdot x = \sum_{i=1}^n a_i(x)(f x_i)$, where each $a_i(x) \in C$.

Since $(f \cdot x)/f \in [P(G)]^L = [P(G)^L]$, we can find $m_i, k_i \in P(G)^L$, $k_i \neq 0$, such that $(f \cdot x_i)/f = m_i / k_i$ or $f \cdot x_i = f(m_i / k_i)$.

Since G is irreducible, $P(G)$ is an integral domain, and since each $k_i \neq 0$, the product $k = k_1 k_2 \cdots k_n$ is a non-zero element of $P(G)^L$.

We have $(f \cdot x)k = (\sum_{i=1}^n a_i(x)(m_i / k_i))kf = (\sum_{i=1}^n a_i(x)(m_i k_i'))f$, where $k_i' = \prod_{j \neq i} k_j$.

Let Z be the set of zeros of f in G_B, where B is the fully stable subalgebra generated by the elements k_i, m_i, $1 \leq i \leq n$, and f.

For $z \in Z, (f \cdot x)(z)k(z) = ((f \cdot x)k)(z) = \sum_{i=1}^{n} a_i(x)m_i(z)k_i'(z)f(z) = 0$, and hence $f(xz)k(z) = 0$ for all $x \in G$. Since $f \neq 0$, we have $k(z) = 0$ for all $z \in Z$. Thus we have a non-zero element k that vanishes on Z. Since F is assumed to be algebraically closed, it follows from the Hilbert Nullstellensatz that there exists a positive integer m such that $k^m = gf$ for some $g \in B$.

Let $y \in L$. Then $y \cdot (gf) = y \cdot k^m = (y \cdot k)^m = k^m = gf$. Since $y \cdot (gf) = (y \cdot g)(y \cdot f) = (y \cdot g)\gamma(y)f$, we have $gf = (y \cdot g)\gamma(y)f$. Hence $(g - \gamma(y)(y \cdot g))f = 0$, which implies that $g - \gamma(y)(y \cdot g) = 0$, and $y \cdot g = \gamma(y)^{-1}g = \gamma(y)_g^{-1}$ follows.

LEMMA 2.3. *Let L be a closed analytic subgroup of a faithfully representable complex analytic group G, and let L^* denote the algebraic closure of L in $A(G)$. If L is observable in G, then L^* is observable in $A(G)$. Conversely, if L^* is observable in $A(G)$, and if $R(G) \to R(L)$ is surjective, then L is observable in G.*

Proof. Assume that L is observable in G, and let V be a finite-dimensional rational L^*-module. Viewed as an analytic L-module, V is a sub L-module of a finite-dimensional analytic G-module W, which in turn becomes a rational $A(G)$-module. The L^*-module V is a sub L^*-module of the $A(G)$-module W, proving that L^* is observable in $A(G)$.

Assume that L^* is observable in $A(G)$ and that the restriction map $R(G) \to R(L)$ is surjective. Thus L^* may be identified with the pro-affine algebraic group $A(L)$. Thus any given finite-dimensional analytic L-module V is a rational L^*-module, which, in turn, is a sub L^*-module of a rational $A(G)$-module W, and the analytic L-module V is then a sub L-module of the analytic G-module W. This shows that L is observable in G.

THEOREM 2.4. *Let L be a closed analytic subgroup of a faithfully representable complex analytic group G. Then L is observable in G if and only if $[R(G)]^L = [R(G)^L]$ and the restriction map $R(G) \to R(L)$ is surjective.*

Proof. Assume that $[R(G)]^L = [R(G)^L]$ and that the restriction map $R(G) \to R(L)$ is surjective. Since $P(A) = R(G)$, $R(G)^{L^*} = R(G)^L$, and $[R(G)]^{L^*} = [R(G)]^L$, we have $[P(A)^{L^*}] = [P(A)]^{L^*}$, and hence L^* is observable in A by Theorem 2.2, and L is observable in G by Lemma 2.3 above.

Assume that L is observable in G. Since the algebraic subgroup L^* is observable in A by Lemma 2.3, we have $[P(A)^{L^*}] = [P(A)]^{L^*}$ by Theorem

2.2, and hence $[R(G)^L] = [R(G)]^L$ follows. That the restriction map $R(G) \to R(L)$ is surjective follows from Lemma 1.1.

3.

Let L be a closed analytic subgroup of a faithfully representable complex analytic group G. In this section we determine when the restriction may $R(G) \to R(L)$ is surjective. Since G is faithfully representable, G is a semidirect product $G = H \cdot K$, where K is a simply connected normal solvable subgroup and H is a maximal reduction subgroup of G. Let N be the representation radical of G (i.e., the intersection of all kernels of semisimple representation of G). N is the radical of the commutator subgroup G' of G, and it is nilpotent and simply connected. Every representation of G induces a unipotent representation on N. Since any two maximal reductive subgroups of G are conjugates to each other by an element from N, it follows that the subgroup HN contains all maximal reductive subgroups of G. As a connected reductive group, H has a unique irreducible algebraic group structure which is compatible with its analytic group structure, and N is given a unique unipotent algebraic group structure. The subgroup NH has the property that its image under any finite-dimensional analytic linear representation of G is algebraic, and it is characterized as the largest normal subgroup of G with this property. We will call the subgroup NH the *algebraic kernel* of G. We note that N is the unipotent radical of the algebraic group NH and that G/NH is a simply connected abelian group since $NH \supset [G, G]$. Finally we note that $R(G, N) = R(G)$. For further discussion, see [2] and [4].

PROPOSITION 3.1. *Let F be a closed normal complex analytic subgroup of a faithfully representable complex analytic group G, and assume that G/F is isomorphic with a complex vector group. Let N be the representation radical of G, and H a maximal reductive subgroup of G. Then $HN \subset F$, and every analytic representation of F that is unipotent on N is extendable to an analytic representation of G that is unipotent on N. In particular, the restriction map $R(G, N) \to R(F, N)$ is surjective.*

Proof. Since G/F is abelian, the commutator subgroup $G' \subset F$, and hence N (= the radical of G') $\subset F$. Since G is faithfully representable, G is a semidirect product $G = H \cdot K$, where K is a simply connected closed normal subgroup of G. The group $HF/F \cong H(/(H \cap F)$ is a complex vector group on one hand, and on the other hand $H/(H \cap F)$ is a complex torus as a homomorphic image of the reductive group H. Thus HF/F is trivial, and $HN \subset F$ follows. By [4, Lemma 2.1], we may find a simply connected nilpotent analytic subgroup P of K such that $K = PN$

and that P centralizes H. Then $G = PNH = PF$, and it is easy to see that $[P, G] \subset N$. Since G/F is a vector group and $G = PF$, we can find complex one-parameter subgroups $P_1, \ldots, P_r \subset P$ such that G is a successive semidirect product $G = P_r \cdots P_1 \cdot F$. Let $B_0 = F$, and define $B_{i+1} = P_{i+1} \cdot B_i$, $i = 0, 1, 2, \ldots, r-1$. Since the subgroup $[B_{i+1}, \mathrm{rad}(B_i)]$ is a solvable normal analytic subgroup of the commutator subgroup G', $[B_{i+1}, \mathrm{rad}(B_i)]$ is contained in N ($= \mathrm{rad}(G')$), where $\mathrm{rad}(Q)$ for any Lie group denotes the solvable radical of Q. Our assertion would follow as soon as we show that every analytic representation ρ of B_i ($0 \leq i \leq r-1$) that is unipotent on N is extendable to an analytic representation σ of B_{i+1} that is unipotent on N. Now applying [3, Theorem 3.1] to the semidirect product $B_{i+1} = P_{i+1} \cdot B_i$, we see that the representation ρ has a desired extension σ. Now the restriction map $R(B_{i+1}, N) \to R(B_i, N)$ is onto for all i by Lemma 1.1, and putting these maps together we see that $R(G, N) \to R(F, N)$ is onto, proving the second assertion.

COROLLARY 3.2. *Let G, N, and H be as in Proposition 3.1. If B is a closed normal analytic subgroup of G such that G/B is a vector group, then there exist complex one-parameter subgroups $P_1, \ldots, P_s$ of B such that $R(B, N) \cong R(P_S) \otimes \cdots \otimes R(P_1) \otimes P(HN)$.*

Proof. By Proposition 3.1 above, $NH \subset B$, and using the notation in the proof of Proposition 3.1 we have $B = (K \cap B) \cdot HN$. Applying the construction of Proposition 3.1 to $F = HN$, we find one parameter subgroups $P_1, \ldots, P_s$ of $P \cap B$ such that B is a successive semidirect product $B = P_s \ldots P_1 \cdot (HN)$ and that for all $1 \leq i \leq s$ the restriction map

$$R(P_i \cdots P_1 \cdot (HN), N) \to R(P_{i-1} \cdots P_1 \cdot (HN), N)$$

is onto. By Lemma 2.1, $R(P_i \cdots P_1 \cdot (HN), N) \cong R(P_i) \otimes R(P_{i-1} \cdots P_1 \cdot (HN), N)$ for all i, and by putting these isomorphisms together we get $R(B, N) \cong R(P_S) \otimes \cdots \otimes R(P_1) \otimes R(HN, N)$. Since $R(HN, N) = P(HN)$, we have $R(B, N) \cong R(P_S) \otimes \cdots \otimes R(P_1) \otimes P(HN)$.

COROLLARY 3.3. *Let B be a closed normal analytic subgroup of a faithfully representable complex analytic group G. Then the* **C**-*algebra $R(B, N)$ is not finitely generated unless $B = HN$.*

Proof. Since $R(\mathbf{C})$ is *not* finitely generated, the assertion follows from Corollary 3.2.

THEOREM 3.4. *Let L be a closed analytic subgroup of G. Then the following are equivalent*:

(i) *The restriction map $R(G) \to R(L)$ is surjective.*

(ii) *The algebraic kernel of L is exactly the intersection of L and the algebraic kernel of G.*

Proof. Let N and V be the representation radicals of G and L, respectively, and let H and D be maximal reductive subgroups of G and L, respectively, so that NH is the algebraic kernel of G and VD is the algebraic kernel of L. With this notation, our assertion amounts to showing that $R(G) \to R(L)$ is onto if and only if $L \cap HN = VD$. We let M denote the subgroup $L \cap NH$.

Suppose that the restriction may $R(G) \to R(L)$ is surjective, and we want to show $M = VD$. We first note that $VD \subset HN$. To see this, let φ be any finite-dimensional analytic representation of G. Then $\varphi(HN)$ is algebraic. The restriction φ_L is also an analytic representation of L, and hence $\varphi_L(VD) = \varphi(VD)$ is also algebraic. Now $VDHN$ is a normal analytic subgroup of G with $\varphi(VDHN) = \varphi(VD)\varphi(HN)$ algebraic. Since HN is the largest such, $VDHN \subset HN$ or $VD \subset HN$, and hence $VD \subset M$. To show $VD \supset M$, we first note that M is topologically connected. In fact, since $L/M \cong LHN/HN$ and since the latter group is a vector group, it follows that M is connected. Since M is a closed normal subgroup of L such that L/M is a vector group, it follows from Proposition 3.1 that the restriction map $R(L) \to R(M, V)$ is surjective, and hence by this together with the surjectivity of $R(G) \to R(L)$ we see that $R(G) \to R(M, V)$ is surjective. Consequently, $R(NH, N) \to R(M, V)$ is surjective. Since $R(NH, N) = P(NH)$ is finitely generated, we see that $R(M, V)$ is finitely generated. However, by Corollary 3.3, $R(M)$ is not finitely generated, unless the vector group M/VD is trivial. Hence we must have $M = VD$.

Assume $M = VD$. Since LNH is a normal analytic subgroup of G such that G/LNH is a complex vector group, the restriction map $R(G) \to R(LNH, N)$ is surjective by Proposition 3.1. Thus to show that the map $R(G) \to R(L)$ is surjective, it is enough to show that $R(LNH, N) \to R(L)$ is surjective. Since $L/M = L/VD$ ($\cong LHN/HN$) is a vector group, we apply Corollary 3.2 to find complex one-parameter subgroups $Q_1, \ldots, Q_s$ of L such that L is a successive semidirect product $L = Q_s \ldots Q_1 \cdot (VD)$ and that

$$R(L) = R(Q_s) \otimes R(Q_1) \otimes \cdots \otimes R(Q_1) \otimes P(VD).$$

Similarly, LHN is a successive semidirect product $LHN = Q_s \cdots Q_1 \cdot (HN)$, and $R(LNH, N) = R(Q_s) \otimes R(Q_1) \otimes \cdots \otimes R(Q_1) \otimes P(HN)$. Since $M = VD$ is an algebraic subgroup of the algebraic group NH, the restriction map $P(VD) \to P(NH)$ is onto, and it follows that the canonical map $R(LNH, N) = R(Q_s) \otimes R(Q_1) \otimes \cdots \otimes R(Q_1) \otimes P(HN) \to R(L) = R(Q_s) \otimes R(Q_1) \otimes \cdots \otimes R(Q_1) \otimes P(VD)$ is onto.

4.

For a subset B of an algebraic linear group, $B^{\#}$ will denote the Zariski closure of B. We will continue to use L^* to denote the algebraic closure of L in $A(G)$, where G is identified with its canonical image in $A(G)$.

LEMMA 4.1. *Let L be a closed analytic subgroup of a complex analytic group G. If there is a faithful finite-dimensional analytic representation* ρ: $G \to GL(V)$ *such that* $\rho(L)^{\#} \cap \rho(G) = \rho(L)$, *then* $L^* \cap G = L$.

Proof. The representation ρ extends to a rational representation ρ': $A \to GL(V)$. Let $x \in L^* \cap G$. Then $\rho(x) = \rho'(x) \in \rho'(L^*) = \rho(L)^{\#}$. Thus $\rho(x) \in \rho(L)^{\#} \cap \rho(G) = \rho(L)$ and hence $x \in L$ because ρ is faithful. Thus $L^* \cap G \subseteq L$, and $L^* \cap G = L$ follows.

LEMMA 4.2. *Let L be a closed analytic subgroup of a faithfully representable complex analytic group G, and assume that the subgroup* $L \cap NH$ *is an algebraic subgroup of NH. Then there exists a finite-dimensional faithful analytic representation* ρ *of G such that* $\rho(L)^{\#} \cap \rho(G) = \rho(L)$.

Proof. Choose any finite-dimensional analytic representation φ of G with its kernel equal to NH such that $\varphi(L)$ is an algebraic subgroup of the general linear group of the representation space of φ. (To get such a representation, choose a faithful unipotent finite-dimensional representation of the vector group G/NH and compose it with the canonical morphism $G \to G/NH$.) Suppose we already have a finite-dimensional faithful analytic representation ψ of G such that $\psi(L)^{\#} \cap \psi(NH) \subseteq \psi(L)$. Then the representation $\rho = \varphi \times \psi$ is a desired representation (that is, $\rho(L)^{\#} \cap \rho(G) = \rho(L)$). In fact, let $(x, y) \in \rho(L)^{\#} \cap \rho(G)$. Thus $(x, y) = \rho(g) = (\varphi(g), \psi(g))$ for some $g \in G$. We will show that $g \in L$. Note that $(x, y) \in \rho(L)^{\#} \subseteq \varphi(L)^{\#} \times \psi(L)^{\#} = \varphi(L) \times \psi(L)^{\#}$. Thus $x = \varphi(g) \in \varphi(L)$ implies $g \in \varphi^{-1}(\varphi(L)) = LNH$. We write $g = znh$ with $(z, n, h) \in L \times N \times H$. Then $z^{-1}g = nh$, and $\psi(z^{-1}g) = \psi(z)^{-1}\psi(g) = \psi(z)^{-1}y \in \psi(L)^{\#}$. Thus $\psi(z^{-1}g) = \psi(nh) \in \psi(L)^{\#} \cap \psi(NH) \subseteq \psi(L)$. Since ψ is faithful, $z^{-1}g \in L$, and $g \in L$ follows.

By what we have shown above, our proof is reduced to showing that there exists a finite-dimensional faithful analytic representation ψ of G such that $\psi(L)^{\#} \cap NH = \psi(L)$. Since G is faithfully representable, we may assume that G is a closed subgroup of a general linear group, say $GL(V)$. Since $L/(NH \cap L)$ is isomorphic with the subgroup LNH/NH of the vector group G/NH, there exist one-parameter subgroups $P_1, P_2, \ldots, P_n$ of G such that G and L are successive semidirect products

$$G = (NH)P_1P_2 \cdots P_n, \; L = (NH \cap L)P_1P_2 \cdots P_r, \qquad r \le n.$$

Since the commutator subgroup $[G,G] \subseteq NH$, we have, for $1 \le i,\ j \le n$,

$$[P_i^{\#}, P_j^{\#}] = [P_i, P_j]^{\#} \subseteq (NH)^{\#} = NH.$$

This shows that, for any subset $\{i_1, i_2, \dots, i_m\}$ of $\{1, 2, \dots, n\}$, the subgroup $NHP_{i_1}^{\#} \cdots P_{i_m}^{\#}$ is the Zariski closure of $NHP_{i_1} \cdots P_{i_m}$, and each $P_j^{\#}$ normalizes $NHP_{i_1}^{\#} \cdots P_{i_m}^{\#}$. In particular, each $P_j^{\#}$ normalizes the subgroup $NHP_1^{\#} \cdots P_{j-1}^{\#}$ of $GL(V)$. We take successive (external) semidirect products $NH \cdot P_1^{\#}, NH \cdot P_1^{\#} \cdot P_2^{\#}, \dots, NH \cdot P_1^{\#} \cdot P_2^{\#} \cdots P_n^{\#}$. The canonical morphism $\psi\colon G = NHP_1P_2 \cdots P_n \to NH \cdot P_1^{\#} \cdot P_2^{\#} \cdots P_n^{\#}$ is a monomorphism, and, since $NH \cap L$ is Zariski closed in $GL(V)$, we have $\psi(L)^{\#} = (NH \cap L) \cdot P_1^{\#} \cdot P_2^{\#} \cdots P_r^{\#}$. Now $\psi(L)^{\#} \cap \psi(G) = (NH \cap L) \cdot P_1^{\#} \cdot P_2^{\#} \cdots P_r^{\#} \cap (NH) \cdot P_1^{\#} \cdot P_2^{\#} \cdots P_n^{\#} = (NH \cap L) \cdot P_1 \cdot P_2 \cdots P_r = \psi(L)$, proving our lemma.

Let L be a closed subgroup of a linear complex algebraic group G. A finite-dimensional analytic representation $\rho\colon L \to GL(V)$ of L is called *rational* if it is the restriction to L of a rational representation $\rho'\colon K \to GL(V)$ of some algebraic subgroup K of G that contains L.

Let L be a linear complex analytic group, and as usual let $L^{\#}$ denote the Zariski closure of L. The image S of the restriction map $P(L^{\#}) \to R(L)$ is a finitely generated fully stable subalgebra of $R(L)$, and since L is algebraically dense in $L^{\#}$, the map $f \to f|_L$ sends $P(L^{\#})$ isomorphically onto S. We may identify $P(L^{\#})$ with S and hence $L^{\#}$ with the group A_S of all proper automorphisms of S, and under this identification the inclusion $L \to L^{\#}$ becomes the canonical map $L \to A_S$. Using this notation, we have

LEMMA 4.3. *A finite-dimensional analytic representation ρ of L is rational if and only if $[\rho] \subseteq S$.*

Proof. Assume ρ is rational, and let $\rho^{\#}$ be a rational representation of $L^{\#}$ such that $\rho^{\#}|L = \rho$. Then we have $[\rho] = [\rho^{\#}] \subseteq S$.

Conversely, assume $[\rho] \subseteq S$. By [2, Proposition 2.5], $[\rho]$ is stable under $A_S = L^{\#}$, and this makes $[\rho]$ a rational $L^{\#}$-module. Now let V be the representation space of ρ. The L-module V may be viewed as an L-submodule of a finite direct sum $W = [\rho] \oplus \cdots \oplus [\rho]$ of the rational $L^{\#}$-module $[\rho]$ by [2, Proposition 2.3]. Our assertion is then to show that V is $L^{\#}$-stable. Choose a basis $v_1, v_2, \dots, v_n$ of the linear space V and enlarge this to a basis $v_1, v_2, \dots, v_m$ $(m \ge n)$ of the linear space W. For each $y \in L^{\#}$, we write

$$y \cdot v_i = a_{1i}(y)v_1 + \cdots + a_{mi}(y)v_m, \qquad \text{where } a_{ki}(y) \in C.$$

Since W is a rational $L^{\#}$-module, the maps $y \to a_{ki}(y)$ $(1 \le k, i \le m)$ are all rational functions on $L^{\#}$. Since V is L-stable, $a_{ki} = 0$ on L (and hence on $L^{\#}$) for all k, i with $n + 1 \le k \le m$, $1 \le i \le n$. This shows that V is $L^{\#}$-stable.

Now let H be a maximal reductive subgroup of a faithfully representable complex analytic group G and let N be the representation kernel of G.

LEMMA 4.4. *Let L be a closed analytic subgroup of G. If every finite-dimensional analytic representation of L is extendable to a finite-dimensional analytic representation of LHN that is unipotent on N, then L is observable in G.*

Proof. Since G/LHN is a complex vector group, the assertion follows from Proposition 3.1.

Let L be a closed complex analytic subgroup of a faithfully representable complex analytic group G, and let L^* be the Zariski closure of L in the proaffine algebraic group $A(G)$. For any finitely generated fully stable subalgebra S of $R(G)$, A_S is an algebraic group with its polynomial algebra S. Since $[S]^{L^*}$ separates the points of the coset variety $L_S^* \backslash A_S$, and since $R(G)$ is a union of finitely generated fully stable subalgebras of $R(G)$, it follows that $[R(G)]^{L^*}$ separates the points of $L^* \backslash G^*$.

Now we are ready to prove our main result.

THEOREM 4.5. *Let G be a faithfully representable complex analytic group, and let L be a closed analytic subgroup of G. Then L is observable in G if and only if L satisfies the following conditions*:

(i) *The algebraic kernel of L is the intersection of L and the algebraic kernel of G.*

(ii) *The algebraic kernel of L is observable in the algebraic kernel of G (in the category of algebraic groups).*

Proof. Suppose that L is observable in G. (i) follows from Theorem 2.4 and Theorem 3.4. Now we prove (ii). By Lemma 4.1 and Lemma 4.2, $L^* \cap G = L$. Since L is observable in $G, [R(G)]^L = [R(G)^L]$ by Theorem 2.4, and since $[R(G)]^{L^*}$ separates the points of $L^* \backslash G^*$ by the remark preceding the theorem above, $[R(G)^L] = [R(G)]^L$ $(= [R(G)]^{L^*})$ separates the points of $L^* \backslash G^*$ and consequently $R(G)^L$ separates the points of $L^* \backslash G^*$. In particular, $R(G)^L$ separates the points of $L^* \backslash L^*G \cong (L^* \cap G) \backslash G = L \backslash G$.

Using [5, Theorem 6.1], we can find a finitely generated fully stable subalgebra S of $R(G)$ which separates the points of G such that S^L separates the points of $L \backslash G$. We identify G with its image G_S of G under the canonical monomorphism $G \to A_S$. Then A_S is the Zariski closure $G^{\#}$ of G, and since S^L separates the points of $L \setminus G$, the fixer of S^L in G is equal to L. It follows that $L^{\#} \cap G = L$.

Let H be a maximal reductive subgroup of G, and let N be the representation radical of G, so that NH is the algebraic kernel of G. Now $L^{\#} \cap NH = L^{\#} \cap G \cap NH = L \cap NH$, and this implies that the subgroup $HN \cap L$ is algebraic in $G^{\#}$. Let $G_1 = L^{\#}HN$ in $G^{\#}$. Since $L^{\#}$ and HN are both algebraic, G_1 is also algebraic and it is a normal subgroup of $G^{\#}$. Since $P(G^{\#})^L$ $(= S^L)$ separates the points of $L \setminus G$, we see that $P(G_1)^L = P(G_1)^{L^{\#}}$ separates the points of $L \setminus LHN$, and hence $P(HN)^{L \cap HN}$ separates the points of $(L \cap HN) \setminus HN$. By [1, Theorem 4], $L \cap HN$ is observable in the algebraic group HN, which proves (ii).

Now assume (i) and (ii), and we want to show that L is observable in G. For this, it is enough to show that the hypothesis of Lemma 4.4 is satisfied. By Lemma 4.2, there exists an analytic monomorphism ρ of G into an algebraic group M such that $\rho(G)$ is algebraically dense in M such that $\rho(L)^{\#} \cap \rho(G) = \rho(L)$. We identify G with $\rho(G)$. Then $L^{\#} \cap G = L$. Now $L^{\#} \cap HN = L^{\#} \cap G \cap HN = L \cap HN$, and $L^{\#} \setminus L^{\#}HN \cong (L^{\#} \cap HN) \setminus HN \cong (L \cap HN) \setminus HN$. Since $(L \cap NH)$ is an observable algebraic subgroup of the algebraic group HN by (i), $(L \cap NH) \setminus HN$ is quasi-affine, and it follows that $L^{\#} \setminus L^{\#}NH$ is quasi-affine. Consequently, $L^{\#}$ is observable in the algebraic subgroup of $L^{\#}NH$ [1, Theorem 4].

Now consider the canonical map $L \setminus LHN \to L^{\#} \setminus L^{\#}HN$, which is induced by the inclusion $L \to L^{\#}$. Since $L^{\#} \cap LHN = L$, the canonical map above is an isomorphism of complex analytic manifolds. We identify $P(L^{\#}HN)$ with a finitely generated fully stable subalgebra S of $R(LHN)$ via the restriction map $P(L^{\#}HN) \to R(LHN)$. Then $L^{\#} = L_S^{\#}$ and $L^{\#}HN = A_S$. Now let ρ be a finite-dimensional analytic representation of L, and let ρ_S denote the corresponding representation of L_S. Assume $[\rho] \subseteq S$. Then, by Lemma 4.3, ρ (or rather ρ_S) is a rational representation. Let ρ_1 be a rational representation of $L_S^{\#}$ such that $\rho_1 | L_S = \rho$. Since $L_S^{\#}$ is observable in $L_S^{\#}H_S N_S$, it follows that there exists a rational representation ρ_2 of $L_S^{\#}H_S N_S$ which extends ρ_1, and the restriction of ρ_2 to $L_S H_S N_S$ provides a desired extension of ρ. Suppose now $[\rho] \not\subset S$, and choose a finitely generated fully stable subalgebra T of $R(LHN)$ that contains S as well as $[\rho]$. Consider the commutative diagram

$$\begin{tikzcd} L \setminus LHN \arrow[r, "\phi_T"] \arrow[dr, "\phi_S"'] & L_T^{\#} \setminus L_T^{\#} H_T N_T \arrow[d] \\ & L_S^{\#} \setminus L_S^{\#} H_S N_S. \end{tikzcd}$$

Since ϕ_S is an isomorphism, it follows that ϕ_T is 1–1, and hence $L_T^{\#} \cap L_T H_T N_T = L_T$. Since $(T^L)'$ separates the points of $L \setminus LHN$, we replace S by T and argue as before to get an extension of ρ which is unipotent on N.

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